

COMPLETION BY DERIVED DOUBLE CENTRALIZER

MARCO PORTA, LIRAN SHAUL AND AMNON YEKUTIELI

ABSTRACT. Let A be a commutative ring, and let \mathfrak{a} be a *weakly proregular* ideal in A . (If A is noetherian then any ideal in it is weakly proregular.) Suppose K is a compact generator of the category of *cohomologically \mathfrak{a} -torsion complexes*. We prove that the *derived double centralizer* of K is isomorphic to the \mathfrak{a} -adic completion of A . The proof relies on the *MGM equivalence* from [PSY]. Our result extends earlier work of Dwyer-Greenlees-Iyengar [DGI] and Efimov [Ef].

0. INTRODUCTION

Let A be a commutative ring. We denote by $D(\text{Mod } A)$ the derived category of A -modules. Given $M \in D(\text{Mod } A)$ we define

$$\text{Ext}_A(M) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D(\text{Mod } A)}(M, M[i]).$$

This is a graded A -algebra with the Yoneda multiplication. The *derived endomorphism DG algebra* of M is a DG A -algebra $B := \text{REnd}_A(M)$, which is well-defined up to quasi-isomorphism. The graded algebra $H(B) := \bigoplus_{i \in \mathbb{Z}} H^i(B)$ has the property that $H(B) \cong \text{Ext}_A(M)$.

Consider the derived category $\tilde{D}(\text{DGM } B)$ of left DG B -modules. We can view M as an object of $\tilde{D}(\text{DGM } B)$, and thus we get the graded A -algebra

$$\text{Ext}_B(M) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\tilde{D}(\text{DGM } B)}(M, M[i]).$$

This is the *derived double centralizer algebra* of M . It is well-defined up to isomorphism. The details on the derived endomorphism DG algebra and the derived double centralizer algebra are worked out in Section 2.

Let \mathfrak{a} be an ideal in A . The \mathfrak{a} -torsion functor $\Gamma_{\mathfrak{a}}$ can be right derived, giving a triangulated functor $\text{R}\Gamma_{\mathfrak{a}}$ from $D(\text{Mod } A)$ to itself. A complex $M \in D(\text{Mod } A)$ is called a *cohomologically \mathfrak{a} -torsion complex* if the canonical morphism $\text{R}\Gamma_{\mathfrak{a}}(M) \rightarrow M$ is an isomorphism. The full triangulated category on the cohomologically torsion complexes is denoted by $D(\text{Mod } A)_{\mathfrak{a}\text{-tor}}$. It is known that when \mathfrak{a} is finitely generated, the category $D(\text{Mod } A)_{\mathfrak{a}\text{-tor}}$ is compactly generated (for instance by the Koszul complex $K(A; \mathfrak{a})$ associated to a finite generating sequence \mathbf{a} of \mathfrak{a}).

The concept of *weakly proregular sequence* in A was introduced in [AJL, Correction] and [Sc]. We recall the definition in Section 1 (it is a bit technical). An

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ideal \mathfrak{a} in A is called *weakly proregular* if it can be generated by a weakly proregular sequence. It is important to note that if A is noetherian, then any finite sequence in it is weakly proregular, and any ideal in A is weakly proregular. But there are some fairly natural non-noetherian examples (see [PSY, Example 3.35]).

Here is our main result (repeated as Theorem 4.2 in the body of the paper).

Theorem 0.1. *Let A be a commutative ring, let \mathfrak{a} be a weakly proregular ideal in A , let K be a compact generator of $\mathbf{D}(\mathbf{Mod} A)_{\mathfrak{a}\text{-tor}}$, and let $B := \mathbf{R}\mathrm{End}_A(K)$. Then $\mathrm{Ext}_B^i(K) = 0$ for all $i \neq 0$, and there is a unique isomorphism of A -algebras $\mathrm{Ext}_B^0(K) \cong \hat{A}$.*

Our result extends earlier work of Dwyer-Greenlees-Iyengar [DGI] and Efimov [Ef]; see Remark 4.9 for a comparison.

Let us say a few words on the proof of Theorem 0.1. We use *derived Morita theory* to find an isomorphism of graded algebras between $\mathrm{Ext}_B(K)$ and $\mathrm{Ext}_A(N)^{\mathrm{op}}$, where $N := \mathbf{R}\Gamma_{\mathfrak{a}}(A)$. The necessary facts about derived Morita theory are recalled in Section 3. We then use *MGM equivalence* (recalled in Section 1) to prove that $\mathrm{Ext}_A(N) \cong \mathrm{Ext}_A(\hat{A}) \cong \hat{A}$.

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1. WEAK PROREGULARITY AND MGM EQUIVALENCE

Let A be a commutative ring, and let \mathfrak{a} be an ideal in it. (We do not assume that A is noetherian or \mathfrak{a} -adically complete.) There are two operations on A -modules associated to this data: the \mathfrak{a} -adic completion and the \mathfrak{a} -torsion. For an A -module M its \mathfrak{a} -adic completion is the A -module $\Lambda_{\mathfrak{a}}(M) = \widehat{M} := \varprojlim_i M/\mathfrak{a}^i M$. An element $m \in M$ is called an \mathfrak{a} -torsion element if $\mathfrak{a}^i m = 0$ for $i \gg 0$. The \mathfrak{a} -torsion elements form the \mathfrak{a} -torsion submodule $\Gamma_{\mathfrak{a}}(M)$ of M .

Let us denote by $\mathbf{Mod} A$ the category of A -modules. So we have additive functors $\Lambda_{\mathfrak{a}}$ and $\Gamma_{\mathfrak{a}}$ from $\mathbf{Mod} A$ to itself. The functor $\Gamma_{\mathfrak{a}}$ is left exact; whereas $\Lambda_{\mathfrak{a}}$ is neither left exact nor right exact. An A -module is called \mathfrak{a} -adically complete if the canonical homomorphism $\tau_M : M \rightarrow \Lambda_{\mathfrak{a}}(M)$ is bijective; and M is \mathfrak{a} -torsion if the canonical homomorphism $\sigma_M : \Gamma_{\mathfrak{a}}(M) \rightarrow M$ is bijective. If the ideal \mathfrak{a} is finitely generated, then the functor $\Lambda_{\mathfrak{a}}$ is idempotent; namely for any module M , its completion $\Lambda_{\mathfrak{a}}(M)$ is \mathfrak{a} -adically complete. (There are counterexamples to that for infinitely generated ideals – see [Ye, Example 1.8].)

The derived category of $\mathbf{Mod} A$ is denoted by $\mathbf{D}(\mathbf{Mod} A)$. The derived functors

$$\mathbf{L}\Lambda_{\mathfrak{a}}, \mathbf{R}\Gamma_{\mathfrak{a}} : \mathbf{D}(\mathbf{Mod} A) \rightarrow \mathbf{D}(\mathbf{Mod} A)$$

exist. The left derived functor $\mathbf{L}\Lambda_{\mathfrak{a}}$ is constructed using K-flat resolutions, and the right derived functor $\mathbf{R}\Gamma_{\mathfrak{a}}$ is constructed using K-injective resolutions. This means that for any K-flat complex P , the canonical morphism $\xi_P : \mathbf{L}\Lambda_{\mathfrak{a}}(P) \rightarrow \Lambda_{\mathfrak{a}}(P)$ is an isomorphism; and for any K-injective complex I , the canonical morphism $\xi_I : \Gamma_{\mathfrak{a}}(I) \rightarrow \mathbf{R}\Gamma_{\mathfrak{a}}(I)$ is an isomorphism.

A complex $M \in \mathbf{D}(\mathbf{Mod} A)$ is called a *cohomologically \mathfrak{a} -torsion complex* if the canonical morphism $\sigma_M^{\mathbf{R}} : \mathbf{R}\Gamma_{\mathfrak{a}}(M) \rightarrow M$ is an isomorphism. The complex M is called a *cohomologically \mathfrak{a} -adically complete complex* if the canonical morphism $\tau_M^{\mathbf{L}} : M \rightarrow \mathbf{L}\Lambda_{\mathfrak{a}}(M)$ is an isomorphism. We denote by $\mathbf{D}(\mathbf{Mod} A)_{\mathfrak{a}\text{-tor}}$ and

$D(\text{Mod } A)_{\mathfrak{a}\text{-com}}$ the full subcategories of $D(\text{Mod } A)$ consisting of cohomologically \mathfrak{a} -torsion complexes and cohomologically \mathfrak{a} -adically complete complexes, respectively. These are triangulated subcategories.

Very little can be said about the functors $L\Lambda_{\mathfrak{a}}$ and $R\Gamma_{\mathfrak{a}}$, and about the corresponding triangulated categories $D(\text{Mod } A)_{\mathfrak{a}\text{-tor}}$ and $D(\text{Mod } A)_{\mathfrak{a}\text{-com}}$, in general. However we know a lot when the ideal \mathfrak{a} is *weakly proregular*.

Before defining weak proregularity we have to talk about *Koszul complexes*. Recall that for an element $a \in A$ the Koszul complex is

$$K(A; a) := (\cdots \rightarrow 0 \rightarrow A \xrightarrow{a} A \rightarrow 0 \rightarrow \cdots),$$

concentrated in degrees -1 and 0 . Given a finite sequence $\mathbf{a} = (a_1, \dots, a_n)$ of elements in A , the Koszul complex associated to this sequence is

$$K(A; \mathbf{a}) := K(A; a_1) \otimes_A \cdots \otimes_A K(A; a_n).$$

This is a complex of finitely generated free A -modules, concentrated in degrees $-n, \dots, 0$. There is a canonical isomorphism of A -modules $H^0(K(A; \mathbf{a})) \cong A/(\mathbf{a})$, where (\mathbf{a}) is the ideal generated by the sequence \mathbf{a} .

For any $i \geq 1$ let $\mathbf{a}^i := (a_1^i, \dots, a_n^i)$. If $j \geq i$ then there is a canonical homomorphism of complexes $p_{j,i} : K(A; \mathbf{a}^j) \rightarrow K(A; \mathbf{a}^i)$, which in H^0 corresponds to the surjection $A/(\mathbf{a}^j) \rightarrow A/(\mathbf{a}^i)$. Thus for every $k \in \mathbb{Z}$ we get an inverse system of A -modules

$$(1.1) \quad \{H^k(K(A; \mathbf{a}^i))\}_{i \in \mathbb{N}},$$

with transition homomorphisms

$$H^k(p_{j,i}) : H^k(K(A; \mathbf{a}^j)) \rightarrow H^k(K(A; \mathbf{a}^i)).$$

Of course for $k = 0$ the inverse limit equals the (\mathbf{a}) -adic completion of A . What turns out to be crucial is the behavior of this inverse system for $k < 0$. For more details please see [PSY, Section 3].

An inverse system $\{M_i\}_{i \in \mathbb{N}}$ of abelian groups, with transition maps $p_{j,i} : M_j \rightarrow M_i$, is called *pro-zero* if for every i there exists $j \geq i$ such that $p_{j,i}$ is zero.

Definition 1.2. (1) Let \mathbf{a} be a finite sequence in A . The sequence \mathbf{a} is called a *weakly proregular sequence* if for every $k \leq -1$ the inverse system (1.1) is pro-zero.

(2) An ideal \mathfrak{a} in A is called a *weakly proregular ideal* if it is generated by some weakly proregular sequence.

The etymology and history of related concepts are explained in [AJL], [AJL, Correction] and [Sc]. If \mathbf{a} is a regular sequence, then it is weakly proregular. More important is the following result of Schenzel.

Theorem 1.3 ([Sc]). *If A is noetherian, then every finite sequence in A is weakly proregular, and hence every ideal in A is weakly proregular.*

Here is another useful fact.

Theorem 1.4 ([PSY]). *Let \mathfrak{a} be a weakly proregular ideal in a ring A . Then any finite sequence that generates \mathfrak{a} is weakly proregular.*

There are interesting examples of weakly proregular ideals in non-noetherian rings (see [PSY, Example 3.35]).

We shall need the following theorem.

Theorem 1.5 (MGM Equivalence, [PSY]). *Let A be a commutative ring, and \mathfrak{a} a weakly proregular ideal in it.*

- (1) *For any $M \in \mathbf{D}(\mathbf{Mod} A)$ one has $\mathbf{R}\Gamma_{\mathfrak{a}}(M) \in \mathbf{D}(\mathbf{Mod} A)_{\mathfrak{a}\text{-tor}}$ and $\mathbf{L}\Lambda_{\mathfrak{a}}(M) \in \mathbf{D}(\mathbf{Mod} A)_{\mathfrak{a}\text{-com}}$.*
- (2) *The functor*

$$\mathbf{R}\Gamma_{\mathfrak{a}} : \mathbf{D}(\mathbf{Mod} A)_{\mathfrak{a}\text{-com}} \rightarrow \mathbf{D}(\mathbf{Mod} A)_{\mathfrak{a}\text{-tor}}$$

is an equivalence, with quasi-inverse $\mathbf{L}\Lambda_{\mathfrak{a}}$.

We should mention that results related to Theorem 1.5 appeared previously in [AJL, Sc, DG].

2. THE DERIVED ENDOMORPHISM ALGEBRA

Let \mathbb{K} be a commutative ring, and let $M = \bigoplus_{i \in \mathbb{Z}} M^i$ and $N = \bigoplus_{i \in \mathbb{Z}} N^i$ be DG A -modules. We denote by $\mathrm{Hom}_{\mathbb{K}}(M, N)^i$ the set of \mathbb{K} -linear homomorphisms $\phi : M \rightarrow N$ of degree i . We get a DG \mathbb{K} -module

$$\mathrm{Hom}_{\mathbb{K}}(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{K}}(M, N)^i$$

with differential

$$(2.1) \quad d(\phi) := d_N \circ \phi - (-1)^i \phi \circ d_M$$

for $\phi \in \mathrm{Hom}_{\mathbb{K}}(M, N)^i$.

Now consider an associative unital noncommutative DG \mathbb{K} -algebra $A = \bigoplus_{i \in \mathbb{Z}} A^i$. Suppose M and N are left DG A -modules. A homomorphism $\phi \in \mathrm{Hom}_{\mathbb{K}}(M, N)^i$ is A -linear (in the graded sense) if $\phi(a \cdot m) = (-1)^{ij} a \cdot \phi(m)$ for all $a \in A^j$ and $m \in M$. The set of all such homomorphisms is denoted by $\mathrm{Hom}_A(M, N)^i$. In this way we obtain a DG \mathbb{K} -module $\mathrm{Hom}_A(M, N)$, which is a DG submodule of $\mathrm{Hom}_{\mathbb{K}}(M, N)$.

The category of left DG A -modules is denoted by $\mathbf{DGM} A$. The set of morphisms $\mathrm{Hom}_{\mathbf{DGM} A}(M, N)$ is precisely the set of 0-cocycles in the DG \mathbb{K} -module $\mathrm{Hom}_A(M, N)$. Note that $\mathbf{DGM} A$ is an abelian category.

For a DG A -module M there is a noncommutative DG \mathbb{K} -algebra $\mathrm{End}_A(M) := \mathrm{Hom}_A(M, M)$. Since the left actions of A and $\mathrm{End}_A(M)$ on M commute, we see that M is a left DG module over the DG algebra $A \otimes_{\mathbb{K}} \mathrm{End}_A(M)$.

For a DG A -module $M = \bigoplus_i M^i$ and $j \in \mathbb{Z}$, the j -th shift of M is the DG A -module $M[j]$ defined as follows. The i -th homogeneous component is $(M[j])^i := M^{i+j}$. The action of A on $M[j]$ is $a \cdot_{[j]} m := (-1)^{ij} a \cdot m$ for $a \in A^i$ and $m \in M$. The differential is $d_{M[j]} := (-1)^j d_M$. In this way the shift $M \mapsto M[j]$ becomes an automorphism of the category $\mathbf{DGM} A$.

Given an A -linear homomorphism $\phi : M \rightarrow N$ of degree i , there is an induced A -linear homomorphism

$$(2.2) \quad \phi[j] := (-1)^{ij} \phi : M[j] \rightarrow N[j],$$

also of degree i . This determines an isomorphism of DG \mathbb{K} -modules

$$\mathrm{Hom}_A(M, N) \xrightarrow{\cong} \mathrm{Hom}_A(M[j], N[j]).$$

When $N = M$ we get a canonical isomorphism of DG \mathbb{K} -algebras

$$(2.3) \quad \mathrm{End}_A(M) \xrightarrow{\cong} \mathrm{End}_A(M[j]),$$

sending $\phi \in \mathrm{End}_A(M)^i$ to $\phi[j] = (-1)^{ij} \phi \in \mathrm{End}_A(M[j])^i$.

The homotopy category of $\mathrm{DGM} A$ is $\tilde{K}(\mathrm{DGM} A)$, and the derived category (got-ten by inverting the quasi-isomorphisms in the homotopy category) is $\tilde{D}(\mathrm{DGM} A)$. All these categories are \mathbb{K} -linear. If A happens to be a ring (i.e. $A^i = 0$ for $i \neq 0$) then $\tilde{D}(\mathrm{DGM} A) = D(\mathrm{Mod} A)$, the usual derived category of left A -modules.

Let A^\natural be the graded algebra gotten from A by forgetting the differential; and the same for $\mathrm{DG} A$ -modules. Recall that a $\mathrm{DG} A$ -module P is called *semi-free* if there is a subset $X \subset P$ consisting of nonzero homogeneous elements, and an exhaustive non-negative increasing filtration $\{F_i(X)\}_{i \in \mathbb{Z}}$ of X by subsets (i.e. $F_{-1}(X) = \emptyset$ and $X = \bigcup F_i(X)$), such that P^\natural is a free graded A^\natural -module with basis X ; and for every i one has $d(X_i) \subset F_{i-1}(P)$, where $F_i(P) := \sum_{x \in F_i(X)} Ax \subset P$. The set X is called a *semi-basis* of P . Any $M \in \mathrm{DGM} A$ admits a quasi-isomorphism $P \rightarrow M$ with P semi-free. A $\mathrm{DG} A$ -module Q is K -projective if and only if it is homotopy equivalent to a semi-free DG module P .

Example 2.4. If A is a ring, then any bounded above complex P of free A -modules is a semi-free $\mathrm{DG} A$ -module. Indeed, let $j_0 := \sup \{j \in \mathbb{Z} \mid P^j \neq 0\}$ (we assume $P \neq 0$). Choose a basis Y_j for the free module P^j , $j \leq j_0$. Define $X := \bigcup_j Y_j$ and $F_i(X) := \bigcup_{j \geq j_0 - i} Y_j$. Then X is a semi-basis for P .

Let $\tilde{K}(\mathrm{DGM} A)_{\mathrm{sf}}$ be the full subcategory of $\tilde{K}(\mathrm{DGM} A)$ consisting of semi-free DG modules. This is a triangulated category. The canonical functor

$$(2.5) \quad \mathrm{En} : \tilde{K}(\mathrm{DGM} A)_{\mathrm{sf}} \rightarrow \tilde{D}(\mathrm{DGM} A)$$

is an equivalence of triangulated categories. See [Sp, BN, Ke, YZ] for details. (The name “En” stands for “enhancement”.)

Suppose B is another DG algebra, and $f : A \rightarrow B$ is a homomorphism of DG algebras. There is an exact functor

$$\mathrm{rest}_{B/A} = \mathrm{rest}_f : \mathrm{DGM} B \rightarrow \mathrm{DGM} A$$

called restriction of scalars (a forgetful functor). It passes to a triangulated functor

$$(2.6) \quad \mathrm{rest}_{B/A} = \mathrm{rest}_f : \tilde{D}(\mathrm{DGM} B) \rightarrow \tilde{D}(\mathrm{DGM} A).$$

In case f is a quasi-isomorphism, then (2.6) is an equivalence (see [YZ, Proposition 1.4]).

Definition 2.7. Let M be a $\mathrm{DG} A$ -module. Define

$$\mathrm{Ext}_A^i(M) := \mathrm{Hom}_{\tilde{D}(\mathrm{DGM} A)}(M, M[i])$$

for any $i \in \mathbb{Z}$. Then $\mathrm{Ext}_A(M) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}_A^i(M)$ is a graded \mathbb{K} -algebra with the Yoneda multiplication (i.e. composition of morphisms in $\tilde{D}(\mathrm{DGM} A)$).

Suppose we are given a $\mathrm{DG} A$ -module P . Let $B := \mathrm{End}_A(P)$ be the algebra of graded A -linear endomorphisms of P . This is a $\mathrm{DG} \mathbb{K}$ -algebra, with differential as in (2.1); and P is a left $\mathrm{DG} B$ -module.

Definition 2.8. Given a $\mathrm{DG} A$ -module M , choose any semi-free resolution $P \rightarrow K$. The *derived endomorphism algebra* of K is the $\mathrm{DG} \mathbb{K}$ -algebra $\mathrm{REnd}_A(M) := \mathrm{End}_A(P)$.

Here is the relation between Definitions 2.7 and 2.8.

Proposition 2.9. *Let $B := \mathrm{REnd}_A(M)$ be the derived endomorphism algebra of M , as in Definition 2.8, constructed using a semi-free resolution $P \rightarrow M$. There is a canonical isomorphism of graded \mathbb{K} -algebras $\mathrm{Ext}_A(M) \cong H(B)$.*

Proof. This is immediate from the equivalence (2.5), since

$$H^i(B) = \mathrm{Hom}_{\tilde{\mathbf{K}}(\mathrm{DGM}_A)}(P, P[i]).$$

□

The dependence of the DG algebra $\mathrm{REnd}_A(M)$ on the resolution $P \rightarrow M$ is explained in the next proposition.

Proposition 2.10. *Let M be a DG A -module, and let $P \rightarrow M$ and $P' \rightarrow M$ be semi-free resolutions. Define $B := \mathrm{End}_A(P)$ and $B' := \mathrm{End}_A(P')$. Then there is a DG \mathbb{K} -algebra B'' , and a DG B'' -module P'' , with DG \mathbb{K} -algebra quasi-isomorphisms $B'' \rightarrow B$ and $B'' \rightarrow B'$, and with isomorphisms*

$$\mathrm{rest}_{B''/B}(P) \cong P'' \cong \mathrm{rest}_{B''/B'}(P')$$

in $\tilde{\mathbf{D}}(\mathrm{DGM } B'')$.

Proof. Choose a quasi-isomorphism $\phi : P' \rightarrow P$ in $\mathrm{DGM } A$ lifting the given quasi-isomorphisms to M . Let $L := \mathrm{cone}(\phi) \in \mathrm{DGM } A$, the mapping cone of ϕ . So as graded A -module $L = P \oplus P'[1] = \begin{bmatrix} P \\ P'[1] \end{bmatrix}$; and the differential is $d_L = \begin{bmatrix} d_P & \phi \\ 0 & d_{P'[1]} \end{bmatrix}$, where ϕ is viewed as a degree 1 homomorphism $P'[1] \rightarrow P$. Of course L is an acyclic DG module.

Take $Q := \mathrm{Hom}_A(P'[1], P)$, and let B'' be the triangular matrix DG algebra $B'' := \begin{bmatrix} B & Q \\ 0 & B' \end{bmatrix}$ with the obvious matrix multiplication. This makes sense because $B' \cong \mathrm{End}_A(P'[1])$ as DG algebras, using the DG algebra isomorphism (2.3). Note that B'' is a subalgebra of $\mathrm{End}_A(L)$. We make B'' into a DG algebra with differential $d_{B''} := d_{\mathrm{End}_A(L)}|_{B''}$. The projections $B'' \rightarrow B$ and $B'' \rightarrow B'$ on the diagonal entries are DG algebra quasi-isomorphisms, because their kernels are the acyclic complexes $\mathrm{Hom}_A(P'[1], L)$ and $\mathrm{Hom}_A(L, P)$ respectively.

Now $\mathrm{rest}_{B''/B}(P) = \begin{bmatrix} P \\ 0 \end{bmatrix}$ and $\mathrm{rest}_{B''/B'}(P') = \begin{bmatrix} 0 \\ P' \end{bmatrix}$ as DG B'' -modules. Define $P'' := \begin{bmatrix} P \\ 0 \end{bmatrix}$. It remains to find an isomorphism $\chi : P'' \xrightarrow{\sim} \mathrm{rest}_{B''/B'}(P')$ in $\mathbf{D}(B'')$. Consider the exact sequence

$$0 \rightarrow \begin{bmatrix} P \\ 0 \end{bmatrix} \rightarrow L \rightarrow \begin{bmatrix} 0 \\ P'[1] \end{bmatrix} \rightarrow 0$$

in $\mathrm{DGM } B''$. There is an induced distinguished triangle $\begin{bmatrix} 0 \\ P' \end{bmatrix} \xrightarrow{\chi} \begin{bmatrix} P \\ 0 \end{bmatrix} \rightarrow L \xrightarrow{\Delta}$ in $\tilde{\mathbf{D}}(\mathrm{DGM } B'')$. But L is acyclic, so χ is an isomorphism. □

Corollary 2.11. *Let $B := \mathrm{REnd}_A(M)$ be the derived endomorphism algebra of M , as in Definition 2.8, constructed using a semi-free resolution $P \rightarrow M$. Then the graded \mathbb{K} -algebra $\mathrm{Ext}_B(P)$ is independent, up to isomorphism, of the resolution $P \rightarrow M$.*

Proof. Let's go back to the proof of Proposition 2.10. Since $B'' \rightarrow B$ is a quasi-isomorphism of DG algebras, it follows that

$$\mathrm{rest}_{B''/B} : \tilde{\mathbf{D}}(\mathrm{DGM } B) \rightarrow \tilde{\mathbf{D}}(\mathrm{DGM } B'')$$

is an equivalence of triangulated categories. Therefore $\text{rest}_{B''/B}$ induces a graded \mathbb{K} -algebra isomorphism $\text{Ext}_B(P) \xrightarrow{\cong} \text{Ext}_{B''}(P'')$. Similarly we get a graded \mathbb{K} -algebra isomorphism $\text{Ext}_{B'}(P') \xrightarrow{\cong} \text{Ext}_{B''}(P'')$. \square

Definition 2.12. Let M be a DG A -module, and let $B := \text{REnd}_A(M)$ be the derived endomorphism algebra of M , as in Definition 2.8, constructed using a semi-free resolution $P \rightarrow M$. The *derived double centralizer* of M is the graded \mathbb{K} -algebra $\text{Ext}_B(M) := \text{Ext}_B(P)$.

The corollary says that the graded algebra $\text{Ext}_B(M)$ is well-defined up to isomorphism.

Remark 2.13. Corollary 2.11 is sufficient for the purposes of this paper (see Theorem 0.1). However it would be nice to know that $\text{Ext}_B(M)$ is well-defined up to a *unique* isomorphism.

Furthermore, we would like to have a better characterization (maybe some sort of axiomatization) of the DG algebra $B = \text{REnd}_A(M)$, and the lift of M to an object of $\tilde{\mathbf{D}}(\text{DGM } B)$. Likewise the DG algebra $\text{REnd}_{\text{REnd}_A(M)}(M)$ should be characterized uniquely in a suitable sense, together with a homomorphism of DG algebras

$$A \rightarrow \text{REnd}_{\text{REnd}_A(M)}(M).$$

Presumably all that should be done in the context of Quillen homotopical algebra.

3. SUPPLEMENT ON DERIVED MORITA THEORY

Derived Morita theory goes back to Rickard's work [Ri], which dealt with rings. Further generalizations can be found in [Ke, BV]. Theorem 3.5 and Corollary 3.8 are “folklore” results, and here we give complete proofs, since we need these results for Section 4.

Suppose A and B are DG \mathbb{K} -algebras, and P is a DG module over $A \otimes_{\mathbb{K}} B^{\text{op}}$. Given a left DG B -module N , there is a left DG A -module $P \otimes_B N$. We get a functor

$$P \otimes_B - : \text{DGM } B \rightarrow \text{DGM } A.$$

The tensor operation respects homotopy equivalences. By restricting it to semi-free DG modules we get a triangulated functor

$$P \otimes_B - : \tilde{\mathbf{K}}(\text{DGM } B)_{\text{sf}} \rightarrow \tilde{\mathbf{K}}(\text{DGM } A).$$

This applies in particular to the case $B := \text{End}_A(P)^{\text{op}}$, since P is automatically a DG $A \otimes_{\mathbb{K}} \text{End}_A(P)$ -module. (Warning: even if P is a semi-free DG A -module, it is usually not even \mathbb{K} -flat over $\text{End}_A(P)$.)

For a DG A -module K we have the derived endomorphism DG algebra $\text{REnd}_A(K)$, which is well-defined up to quasi-isomorphism. See Definition 2.8 and Proposition 2.10.

Proposition 3.1. *Let \mathbf{E} be a full triangulated subcategory of $\tilde{\mathbf{D}}(\text{DGM } A)$, closed under infinite direct sums, and let K be an object of \mathbf{E} . Define $B := \text{REnd}_A(K)^{\text{op}}$. Then there is a \mathbb{K} -linear triangulated functor $G : \tilde{\mathbf{D}}(\text{DGM } B) \rightarrow \mathbf{E}$ with these properties:*

- (i) *G commutes with infinite direct sums, and $G(B) \cong K$.*

- (ii) Let $P \rightarrow K$ be the semi-free resolution used to define B , namely $B = \text{End}_A(P)^{\text{op}}$. Then the functor

$$G \circ \text{En} : \tilde{K}(\text{DGM } B)_{\text{sf}} \rightarrow \tilde{D}(\text{DGM } A)$$

is isomorphic to $P \otimes_B -$.

Moreover, such a functor G is unique up to isomorphism.

Proof. Choose a semi-free resolution $P \rightarrow K$, and define $G(Q) := P \otimes_B Q$ for a semi-free DG B -module Q . The semi-free enhancement (2.5) for the DG algebra B says that G extends, uniquely up to isomorphism, to a triangulated functor $G : \tilde{D}(\text{DGM } B) \rightarrow \mathbf{E}$. This functor G commutes with infinite direct sums, property (ii) holds, and $G(B) = P \otimes_B B \cong P \cong K$. \square

Definition 3.2. Let \mathbf{E} be a full triangulated subcategory of $\tilde{D}(\text{DGM } A)$, closed under infinite direct sums. A DG A -module K is said to be *compact relative to \mathbf{E}* if for any collection $\{N_i\}_{i \in I}$ of objects of \mathbf{E} , the canonical homomorphism

$$\bigoplus_{i \in I} \text{Hom}_{\tilde{D}(\text{DGM } A)}(K, N_i) \rightarrow \text{Hom}_{\tilde{D}(\text{DGM } A)}\left(K, \bigoplus_{i \in I} N_i\right)$$

is bijective.

As usual, if K is itself in \mathbf{E} , then one calls K a *compact object of \mathbf{E}* .

Let P be a DG module over $A \otimes_{\mathbb{K}} B^{\text{op}}$, as above. For any $N \in \text{DGM } A$, we have a DG B -module $\text{Hom}_A(P, N)$. Thus we get a functor

$$\text{Hom}_A(P, -) : \text{DGM } A \rightarrow \text{DGM } B.$$

The functor $\text{Hom}_A(P, -)$ respects homotopies, and hence we get an induced triangulated functor

$$\text{Hom}_A(P, -) : \tilde{K}(\text{DGM } A) \rightarrow \tilde{K}(\text{DGM } B).$$

Proposition 3.3. Let K be a DG A -module, and let $B := \text{REnd}_A(K)^{\text{op}}$. There is a \mathbb{K} -linear triangulated functor

$$F : \tilde{D}(\text{DGM } A) \rightarrow \tilde{D}(\text{DGM } B)$$

with these properties:

- (i) $F(K) \cong B$ in $\tilde{D}(\text{DGM } B)$.
- (ii) Let \mathbf{E} be a full triangulated subcategory of $\tilde{D}(\text{DGM } A)$, closed under infinite direct sums. The functor $F|_{\mathbf{E}} : \mathbf{E} \rightarrow \tilde{D}(\text{DGM } B)$ commutes with infinite direct sums if and only if K is a compact object relative to \mathbf{E} .
- (iii) Let $P \rightarrow K$ be the semi-free resolution used to define B , namely $B = \text{End}_A(P)^{\text{op}}$. Then the functor

$$F \circ \text{En} : \tilde{K}(\text{DGM } A)_{\text{sf}} \rightarrow \tilde{D}(\text{DGM } B)$$

is isomorphic to $\text{Hom}_A(P, -)$.

Moreover, the functor F is unique up to isomorphism.

Proof. Existence of F , and property (iii), are immediate from the equivalence (2.5). Since $K \cong P$ in $\tilde{D}(\text{DGM } A)$ it follows that $F(K) \cong F(P) \cong \text{Hom}_A(P, P) = B$.

It remains to consider property (ii). We know that

$$\text{Hom}_{\tilde{D}(\text{DGM } A)}(K, N[j]) \cong H^j(\text{RHom}_A(K, N)) \cong H^j(F(N)),$$

functorially for $N \in \tilde{\mathbf{D}}(\mathrm{DGM} A)$. So K is compact relative to \mathbf{E} if and only if the functors $H^j \circ F$ commute with direct sums in \mathbf{E} . But that is the same as asking F to commute with direct sums in \mathbf{E} . \square

Lemma 3.4. *Let \mathbf{E} be a triangulated category with infinite direct sums, let $F, G : \tilde{\mathbf{D}}(\mathrm{DGM} A) \rightarrow \mathbf{E}$ be triangulated functors that commute with infinite direct sums, and let $\eta : F \rightarrow G$ be a morphism of triangulated functors. Assume that $\eta_A : F(A) \rightarrow G(A)$ is an isomorphism. Then η is an isomorphism.*

Proof. Suppose we are given a distinguished triangle $M' \rightarrow M \rightarrow M'' \xrightarrow{\Delta}$ in $\tilde{\mathbf{D}}(\mathrm{DGM} A)$, such that two of the three morphisms $\eta_{M'}$, η_M and $\eta_{M''}$ are isomorphisms. Then the third is also an isomorphism.

Since both functors F, G commute with shifts and direct sums, and since η_A is an isomorphism, it follows that η_P is an isomorphism for any free DG A -module P .

Next consider a semi-free DG module P . Choose any semi-basis $X = \bigcup X_j$ of P . This gives rise to a filtration $\{F_j(P)\}_{j \in \mathbb{Z}}$ of P by DG submodules, with $F_{-1}(P) = 0$. For every j we have a distinguished triangle

$$F_{j-1}(P) \xrightarrow{\theta_j} F_j(P) \rightarrow F_j(P)/F_{j-1}(P) \xrightarrow{\Delta}$$

in $\tilde{\mathbf{D}}(\mathrm{DGM} A)$, where $\theta_j : F_{j-1}(P) \rightarrow F_j(P)$ is the inclusion. Since $F_j(P)/F_{j-1}(P)$ is free, by induction we conclude that $\eta_{F_j(P)}$ is an isomorphism for every j . The telescope construction (see [BN, Remark 2.2]) gives a distinguished triangle

$$\bigoplus_{j \in \mathbb{N}} F_j(P) \xrightarrow{\Theta} \bigoplus_{j \in \mathbb{N}} F_j(P) \rightarrow P \xrightarrow{\Delta},$$

with

$$\Theta|_{F_{j-1}(P)} := (\mathrm{id}, -\theta_j) : F_{j-1}(P) \rightarrow F_{j-1}(P) \oplus F_j(P).$$

This shows that η_P is an isomorphism.

Finally, any DG module M admits a quasi-isomorphism $P \rightarrow M$ with P semi-free. Therefore η_M is an isomorphism. \square

Theorem 3.5. *Let \mathbf{E} be a full triangulated subcategory of $\tilde{\mathbf{D}}(\mathrm{DGM} A)$, closed under infinite direct sums, and let K be a compact object of \mathbf{E} . Define $B := \mathrm{REnd}_A(K)^{\mathrm{op}}$. Consider the \mathbb{K} -linear triangulated functors $G : \tilde{\mathbf{D}}(\mathrm{DGM} B) \rightarrow \mathbf{E}$ and $F : \mathbf{E} \rightarrow \tilde{\mathbf{D}}(\mathrm{DGM} B)$ from the previous propositions.*

- (1) *There are morphisms of triangulated functors $\eta : \mathbf{1} \rightarrow F \circ G$ and $\zeta : G \circ F \rightarrow \mathbf{1}$, that make F into a right adjoint of G .*
- (2) *The morphism η is an isomorphism. Hence the functor G is fully faithful.*
- (3) *Let $M \in \mathbf{E}$. Then M is in the essential image of the functor G if and only if the morphism $\zeta_M : (G \circ F)(M) \rightarrow M$ is an isomorphism.*

Proof. Let $P \rightarrow K$ be the semi-free resolution used to construct B ; namely $B = \mathrm{End}_A(P)^{\mathrm{op}}$.

Take any $M \in \mathbf{E}$ and $N \in \tilde{\mathbf{D}}(\mathrm{DGM} B)$. We have to construct a bijection

$$\mathrm{Hom}_{\tilde{\mathbf{D}}(\mathrm{DGM} A)}(G(N), M) \cong \mathrm{Hom}_{\tilde{\mathbf{D}}(\mathrm{DGM} B)}(N, F(M)),$$

which is bifunctorial. Choose a semi-free resolution $Q \rightarrow N$ over B . Since the DG A -module $P \otimes_B Q$ is semi-free, we have a sequence of isomorphisms (of \mathbb{K} -modules)

$$\begin{aligned} \mathrm{Hom}_{\tilde{\mathbf{D}}(\mathrm{DGM}_A)}(G(N), M) &\cong \mathrm{H}^0(\mathrm{RHom}_A(G(N), M)) \\ &\cong \mathrm{H}^0(\mathrm{Hom}_A(P \otimes_B Q, M)) \cong \mathrm{H}^0(\mathrm{Hom}_B(Q, \mathrm{Hom}_A(P, M))) \\ &\cong \mathrm{H}^0(\mathrm{RHom}_B(N, F(M))) \cong \mathrm{Hom}_{\tilde{\mathbf{D}}(\mathrm{DGM}_B)}(N, F(M)). \end{aligned}$$

The only choice made was in the semi-free resolution $Q \rightarrow N$, so all is bifunctorial. The corresponding morphisms $\mathbf{1} \rightarrow F \circ G$ and $G \circ F \rightarrow \mathbf{1}$ are denoted by η and ζ respectively.

We have to prove that the morphism $\eta_N : N \rightarrow (F \circ G)(N)$ in $\tilde{\mathbf{D}}(\mathrm{DGM}_B)$ is an isomorphism. Since the functors $\mathbf{1}$ and $F \circ G$ commute with infinite direct sums, it suffices (by Lemma 3.4) to check for $N = B$. But in this case η_B is the canonical homomorphism of DG B -modules $B \rightarrow \mathrm{Hom}_A(P, P \otimes_B B)$, which is clearly bijective.

It remains to prove part (3). If ζ_M is an isomorphism then trivially M is in the essential image of G . Conversely, assume that $M \cong G(N)$ for some DG B -module N . It is enough to prove that $\zeta_{G(N)}$ is an isomorphism. But under the bijection

$$\mathrm{Hom}_{\tilde{\mathbf{D}}(\mathrm{DGM}_B)}(N, N) \cong \mathrm{Hom}_{\tilde{\mathbf{D}}(\mathrm{DGM}_A)}(G(N), G(N))$$

induced by G , 1_N goes to $\zeta_{G(N)}$. So $\zeta_{G(N)}$ is invertible. \square

Definition 3.6. Let \mathbf{E} be a triangulated category. An object $K \in \mathbf{E}$ is called a *generator* if for any nonzero $M \in \mathbf{E}$ there is some integer i such that $\mathrm{Hom}_{\mathbf{E}}(K, M[i])$ is nonzero.

Remark 3.7. The notion of “generator” above is the weakest among several found in the literature. See [BV] for discussion.

Corollary 3.8. *In the situation of Theorem 3.5, suppose that K is a compact generator of \mathbf{E} . Then the \mathbb{K} -linear triangulated functor $G : \tilde{\mathbf{D}}(\mathrm{DGM}_B) \rightarrow \mathbf{E}$ is an equivalence, with quasi-inverse F .*

Proof. In view of property (2) of Theorem 1.5, all we have to prove is that G is essentially surjective on objects. Take any $L \in \mathbf{E}$, and consider the distinguished triangle $(G \circ F)(L) \xrightarrow{\zeta_L} L \rightarrow M \xrightarrow{\Delta}$ in \mathbf{E} , in which M is the mapping cone of ζ_L . Applying F and using η we get a distinguished triangle $F(L) \xrightarrow{1_{F(L)}} F(L) \rightarrow F(M) \xrightarrow{\Delta}$. So $F(M) = 0$. But $\mathrm{RHom}_A(K, M) \cong F(M)$, and therefore $\mathrm{Hom}_{\mathbf{D}(A)}(K, M[i]) = 0$ for every i . Since K is a generator of \mathbf{E} we get $M = 0$. Hence ζ_L is an isomorphism, and so L is in the essential image of G . \square

Remark 3.9. The proofs above work also for the triangulated category $\mathbf{D}(\mathbf{C})$, where \mathbf{C} is any abelian category with exact infinite direct sums and enough projectives. The changes needed are minor – one needs the K-projective enhancement of $\mathbf{D}(\mathbf{C})$.

Remark 3.10. A result similar to Theorem 3.5 should be true for the derived category $\mathbf{D}(\mathbf{C})$ of a Grothendieck abelian category \mathbf{C} ; for instance $\mathbf{C} := \mathrm{Mod} \mathcal{A}$, where (X, \mathcal{A}) is a ringed space. Here one needs the K-injective enhancement of the triangulated category $\mathbf{D}(\mathbf{C})$. See [KS, Theorem 14.3.1]. The details are more difficult.

4. THE MAIN THEOREM

This is our interpretation of the completion appearing in Efimov's recent paper [Ef], that is attributed to Kontsevich; cf. Remark 4.9 below. Here is the setup for this section: A is a commutative ring, and \mathfrak{a} is a weakly proregular ideal in A . We do not assume that A is noetherian nor \mathfrak{a} -adically complete. Let $\hat{A} := \Lambda_{\mathfrak{a}}(A)$, the \mathfrak{a} -adic completion of A .

Recall the Koszul complex $K(A; \mathfrak{a})$ associated to a finite sequence \mathfrak{a} in A ; see Section 1. It is a bounded complex of free A -modules, and hence it is a semi-free DG A -module. The next result was proved by several authors (see [BN, Proposition 6.1], [LN, Corollary 5.7.1(ii)] and [Ro, Proposition 6.6]).

Proposition 4.1. *Let \mathfrak{a} be a finite sequence that generates \mathfrak{a} . Then the Koszul complex $K(A; \mathfrak{a})$ is a compact generator of $D(\text{Mod } A)_{\mathfrak{a}\text{-tor}}$, in the sense of Definitions 3.2 and 3.6.*

Let K be a compact generator of $D(\text{Mod } A)_{\mathfrak{a}\text{-tor}}$. Choose a semi-free resolution $P \rightarrow K$ over A . If K is already semi-free, then we can take $P = K$. Consider the derived endomorphism DG algebra $B = \text{REnd}_A(K) := \text{End}_A(P)$ as in Definition 2.8, where we take $\mathbb{K} := A$. So B is a noncommutative DG A -algebra. There is the double derived centralizer $\text{Ext}_B(K)$, which is a graded A -algebra (see Definition 2.12). By Corollary 2.11, $\text{Ext}_B(K)$ is independent of the resolution $P \rightarrow M$, up to isomorphism.

Theorem 4.2. *Let A be a commutative ring, let \mathfrak{a} be a weakly proregular ideal in A , let K be a compact generator of $D(\text{Mod } A)_{\mathfrak{a}\text{-tor}}$, and let $B := \text{REnd}_A(K)$. Then $\text{Ext}_B^i(K) = 0$ for all $i \neq 0$, and there is a unique isomorphism of A -algebras $\text{Ext}_B^0(K) \cong \hat{A}$.*

We need a few lemmas first.

Lemma 4.3. *Let K be a compact object of $D(\text{Mod } A)_{\mathfrak{a}\text{-tor}}$. Then K is also compact in $D(\text{Mod } A)$, so it is a perfect complex of A -modules.*

Proof. Choose a finite sequence \mathfrak{a} that generates \mathfrak{a} . By [PSY, Corollary 3.26] there is an isomorphism of functors $\text{R}\Gamma_{\mathfrak{a}} \cong \text{K}_{\infty}^{\vee}(A; \mathfrak{a}) \otimes_A -$, where $\text{K}_{\infty}^{\vee}(A; \mathfrak{a})$ is the infinite dual Koszul complex. Therefore the functor $\text{R}\Gamma_{\mathfrak{a}}$ commutes with direct sums.

Let $M \in D(\text{Mod } A)$, and consider the function

$$\text{Hom}(1, \sigma_M^{\text{R}}) : \text{Hom}_{D(\text{Mod } A)}(K, \text{R}\Gamma_{\mathfrak{a}}(M)) \rightarrow \text{Hom}_{D(\text{Mod } A)}(K, M).$$

Given a morphism $\alpha : K \rightarrow M$ in $D(\text{Mod } A)$ define

$$\beta := \text{R}\Gamma_{\mathfrak{a}}(\alpha) \circ (\sigma_K^{\text{R}})^{-1} : K \rightarrow \text{R}\Gamma_{\mathfrak{a}}(M).$$

Since the functor $\text{R}\Gamma_{\mathfrak{a}}$ is idempotent (Theorem 1.5(1)), the function $\alpha \mapsto \beta$ is an inverse to $\text{Hom}(1, \sigma_M^{\text{R}})$, so the latter is bijective.

Let $\{M_i\}_{i \in I}$ be a collection of objects of $D(\text{Mod } A)$. Due to the fact that K is a compact object of $D(\text{Mod } A)_{\mathfrak{a}\text{-tor}}$, and to the observations above, we get isomorphisms

$$\begin{aligned} \bigoplus_i \text{Hom}_{D(\text{Mod } A)}(K, M_i) &\cong \bigoplus_i \text{Hom}_{D(\text{Mod } A)}(K, \text{R}\Gamma_{\mathfrak{a}}(M_i)) \\ &\cong \text{Hom}_{D(\text{Mod } A)}(K, \bigoplus_i \text{R}\Gamma_{\mathfrak{a}}(M_i)) \cong \text{Hom}_{D(\text{Mod } A)}(K, \text{R}\Gamma_{\mathfrak{a}}(\bigoplus_i M_i)) \\ &\cong \text{Hom}_{D(\text{Mod } A)}(K, \bigoplus_i M_i). \end{aligned}$$

We see that K is also compact in $D(\text{Mod } A)$. \square

Consider the contravariant functor

$$D : D(\text{Mod } B) \rightarrow D(\text{Mod } B^{\text{op}})$$

defined by choosing an injective resolution $A \rightarrow I$ over A , and letting $D := \text{Hom}_A(-, I)$.

Lemma 4.4. *The functor D induces a duality (i.e. a contravariant equivalence) between the full subcategory of $D(\text{Mod } B)$ consisting of objects perfect over A , and the full subcategory of $D(\text{Mod } B^{\text{op}})$ consisting of objects perfect over A .*

Proof. Take $K \in D(\text{Mod } B)$ which is perfect over A . It is enough to show that the canonical homomorphism of DG B -modules

$$(4.5) \quad K \rightarrow (D \circ D)(K) = \text{Hom}_A(\text{Hom}_A(K, I), I)$$

is a quasi-isomorphism. For this we can forget the B -module structure, and just view this as a homomorphism of DG A -modules. Choose a resolution $P \rightarrow K$ where P is a bounded complex of finitely generated projective A -modules. We can replace K with P in equation (4.5); and now it is clear that this is a quasi-isomorphism. \square

Lemma 4.6. *Let M and N be K -flat complexes of A -modules. We write $\widehat{M} := \Lambda_{\mathfrak{a}}(M)$ and $\widehat{N} := \Lambda_{\mathfrak{a}}(N)$.*

- (1) *The morphisms $\xi_M : \text{L}\Lambda_{\mathfrak{a}}(M) \rightarrow \widehat{M}$ and $\tau_M^{\text{L}} : \widehat{M} \rightarrow \text{L}\Lambda_{\mathfrak{a}}(\widehat{M})$ are isomorphisms.*
- (2) *The homomorphism*

$$\text{Hom}(\tau_M, 1) : \text{Hom}_{D(\text{Mod } A)}(\widehat{M}, \widehat{N}) \rightarrow \text{Hom}_{D(\text{Mod } A)}(M, N)$$

is bijective.

Proof. (1) The morphism ξ_M is an isomorphism by [PSY, Proposition 2.6]. By Theorem 1.5(1) the complex $\text{L}\Lambda_{\mathfrak{a}}(M)$ is cohomologically complete; and therefore \widehat{M} is also cohomologically complete. But this mean that τ_M^{L} is an isomorphism.

(2) Take a morphism $\alpha : M \rightarrow N$ in $D(\text{Mod } A)$. By part (1) we know that ξ_M and τ_N^{L} are isomorphisms, so we can define

$$\beta := (\tau_N^{\text{L}})^{-1} \circ \text{L}\Lambda_{\mathfrak{a}}(\alpha) \circ \xi_M^{-1} : \widehat{M} \rightarrow \widehat{N}.$$

The function $\alpha \mapsto \beta$ is an inverse to $\text{Hom}(\tau_M, 1)$. \square

Proof of Theorem 4.2. Let us calculate $\text{Ext}_B(K)$ indirectly. By Lemma 4.3 we know that K is perfect over A . Choose a resolution $P \rightarrow K$ where P is a bounded complex of finitely generated projective A -modules. We can now take $B := \text{End}_A(P)$.

According to Lemma 4.4 we get an isomorphism of graded A -algebras

$$\text{Ext}_B(K) \cong \text{Ext}_{B^{\text{op}}}(D(K))^{\text{op}}.$$

Next we note that

$$D(K) = \text{Hom}_A(K, I) \cong \text{Hom}_A(P, I) \cong \text{Hom}_A(P, A) = F(A)$$

in $D(B^{\text{op}})$. Here

$$(4.7) \quad F : D(\text{Mod } A) \rightarrow \tilde{D}(\text{DGM } B^{\text{op}})$$

is the equivalence from Proposition 3.3. Therefore we get an isomorphism of graded A -algebras $\mathrm{Ext}_{B^{\mathrm{op}}}(D(K)) \cong \mathrm{Ext}_{B^{\mathrm{op}}}(F(A))$.

Let $N := \mathrm{R}\Gamma_{\mathfrak{a}}(A) \in \mathrm{D}(\mathrm{Mod} A)$. We claim that $F(A) \cong F(N)$ in $\tilde{\mathrm{D}}(\mathrm{DGM} B^{\mathrm{op}})$. To see this, we first note that the canonical morphism $N \rightarrow A$ in $\mathrm{D}(\mathrm{Mod} A)$ can be represented by an actual DG module homomorphism $N \rightarrow A$ (say by replacing N with a K-projective resolution of it). Consider the induced homomorphism $\mathrm{Hom}_A(P, N) \rightarrow \mathrm{Hom}_A(P, A)$ of DG B^{op} -modules. Like in the proof of Lemma 4.4, it suffices to show that this is a quasi-isomorphism of DG A -modules. This is true since the canonical morphism $\mathrm{RHom}_A(K, N) \rightarrow \mathrm{RHom}_A(K, A)$ in $\mathrm{D}(A)$ is an isomorphism. We conclude that $\mathrm{Ext}_{B^{\mathrm{op}}}(F(A)) \cong \mathrm{Ext}_{B^{\mathrm{op}}}(F(N))$.

Using the equivalence F of (4.7), and the fact that $\mathrm{D}(\mathrm{Mod} A)_{\mathfrak{a}\text{-tor}}$ is full in $\mathrm{D}(\mathrm{Mod} A)$, we see that F induces an isomorphism of graded A -algebras $\mathrm{Ext}_{B^{\mathrm{op}}}(F(N)) \cong \mathrm{Ext}_A(N)$.

The next step is to use the MGM equivalence. We know that $\mathrm{L}\Lambda_{\mathfrak{a}}(N) \cong \hat{A}$ in $\mathrm{D}(\mathrm{Mod} A)$. And the functor $\mathrm{L}\Lambda_{\mathfrak{a}}$ induces an isomorphism of graded A -algebras $\mathrm{Ext}_A(N) \cong \mathrm{Ext}_A(\hat{A})$.

It remains to analyze the graded A -algebra $\mathrm{Ext}_A(\hat{A})$. By Lemma 4.6 the homomorphism

$$\mathrm{Hom}(\tau_A, 1) : \mathrm{Hom}_{\mathrm{D}(\mathrm{Mod} A)}(\hat{A}, \hat{A}[i]) \rightarrow \mathrm{RHom}_{\mathrm{D}(\mathrm{Mod} A)}(A, \hat{A}[i])$$

is bijective for every i . Therefore $\mathrm{Ext}_A^i(\hat{A}) = 0$ for $i \neq 0$, and the A -algebra homomorphism $\hat{A} \rightarrow \mathrm{Ext}_A^0(\hat{A})$ is bijective. Since the image of A in \hat{A} is a dense subalgebra, it follows that this algebra isomorphism is unique.

Combining all the steps above we see that $\mathrm{Ext}_B^i(K) = 0$ for $i \neq 0$, and there is a unique A -algebra isomorphism $\mathrm{Ext}_B^0(K) \cong \hat{A}^{\mathrm{op}}$. But \hat{A} is commutative, so $\hat{A}^{\mathrm{op}} = \hat{A}$. \square

Remark 4.8. To explain how surprising this theorem is, take the case $K := \mathrm{K}(A; \mathbf{a})$, the Koszul complex associated to a sequence $\mathbf{a} = (a_1, \dots, a_n)$ that generates the ideal \mathfrak{a} . This is a semi-free complex, so we might as well take $P = K$ in the proof above.

As free A -module (forgetting the grading and the differential), we have $K = A^{n^2}$. The grading of K depends on n only (it is an exterior algebra). The differential of K is the only place where the sequence \mathbf{a} enters. Similarly, the DG algebra $B = \mathrm{End}_A(K)$ is a graded matrix algebra over A , of size $n^2 \times n^2$. The differential of B is where \mathbf{a} is expressed.

Forgetting the differentials, i.e. working with the graded module K^{\natural} over the graded algebra A^{\natural} , classical Morita theory tells us that $\mathrm{End}_{B^{\natural}}(K^{\natural}) = A^{\natural}$ as graded A -algebras. Furthermore, K^{\natural} is a projective B^{\natural} -module, so we even have $\mathrm{Ext}_{B^{\natural}}(K^{\natural}) = A^{\natural}$.

However, the theorem tells us that for the DG-module structure of K we have $\mathrm{Ext}_B(K) \cong \hat{A}$. Thus we get a transcendental outcome – the completion \hat{A} – by a homological operation with finite input (basically finite linear algebra over A together with a differential).

Remark 4.9. Our motivation to work on completion by derived double centralizer came from looking at the recent paper [Ef] by Efimov. He proves Theorem 4.2, under the extra assumption that A is a regular noetherian ring (i.e. it has finite global cohomological dimension).

After writing the first version of our paper, we learned that a similar result was proved in [DGI], again under an extra assumption: A is noetherian and $A_0 := A/\mathfrak{a}$ is a regular ring.

Recall that our Theorem 4.2 only requires the ideal \mathfrak{a} to be weakly proregular, and there is no regularity condition (this word has a double meaning here!) on the rings A and A_0 .

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DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY, BE'ER SHEVA 84105, ISRAEL

E-mail address: (PORTA) marcoporta1@libero.it, (SHAUL) shlir@math.bgu.ac.il, (YEKUTIELI) amyekut@math.bgu.ac.il